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Auxiliary fields for treating two-time Lagrangian quantum mechanics

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Abstract. Functional integrals over complex and real auxiliary fields for propagators relating to two-time (memory) actions are constructed. The integrals over fields result from partial replacement of the path variable, via functional integration, in the path integral form of the propagator by a field variable in a way that allows the remaining path integration to be performed, thus leaving a functional integral. Two new cases of actions involving memory, a quadratic and a quartic are treated explicitly. A functional integral identity is also developed. Furthermore, the method of coupling an auxiliary dynamical field for generating memory actions is also employed and formulae have been developed in the case of certain quadratic actions giving the propagator exactly. The method is flexible enough embracing all explicitly known cases of two-time quadratic actions and in addition enabling further cases to be treated. An example of this nature is given explicitly.

1. Introduction

The forces entering the equations of motion of a system derive from a potential provided the dynamics of all other systems interacting with it are coupled to the system's dynamical equations. In such a situation we deal essentially with the dynamics of a closed system possessing potential independent of time, which at a given moment is fully determined by the instantaneous degrees of freedom of the whole system. However, on many occasions attention is focused on a subsystem of a closed system and in such a case the particular system's potential function becomes time dependent or may also manifest a memory effect which gives rise to a two-time action.

In the literature one finds circumstances in which memory in a particular system's Lagrangian results from conditional elimination of the degrees of freedom of the systems that are coupled to the one in question. A frequently encountered case of this nature is in the polaron problem (Feynman 1955, Krivoglaz and Pekar 1957, Osaka 1958, Hellwarth and Platzman 1962, Thornber and Feynman 1970, Thornber 1971). A more general situation of this sort appears in the theory of influence functionals (Feynman and Vernon 1963).

The average propagator of a conduction electron moving in a system of disordered scatterers (amorphous material) leads under certain conditions to a two-time action (Edwards and Guliaev 1962, Edwards 1970a, b). Work involving this action was subsequently pursued by Jones and Lukes (1969), Samathiyakanit (1974), Gross (1977) and Sa Yakamit (1979). Furthermore, such action appears in studies of polymerised matter, but the role of time now taken up by the polymer length (Edwards 1965, 1966, Edwards and Miller 1975).

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Because of the difficulty inherent in evaluating propagators deriving from such actions one is forced to resort to approximations involving at some stage a two-time quadratic action. In the case of such an action the propagator takes the form of a Gaussian path integral which equals a product of a time dependent factor and the exponential of $(\mathrm{i}/\hbar)$ times the classical action. Unlike the case of the single-time quadratic action the time dependent factor cannot be determined by the classical action alone using the Van Vleck (1928) determinant as e.g. in Jones and Papadopoulos (1971). However, as pointed out by Adamowski *et al* (1982) if one deals with certain aspects of statistical mechanics, evaluation of the time dependent factor may not be needed. In general the full expression for the propagator is necessary and an increasing number of methods have been devised (Maheshwàri 1975, Khandekar *et al* 1981, Dahra *et al* 1982) for treating a simple two-time quadratic action used by Bezák (1970), following a solution employing an appropriate auxiliary field (Papadopoulos 1974).

Recently more general quadratic actions have been treated by Adamowski *et al* (1982) making use of functional integration and subsequently by Khandekar *et al* (1983) employing path integration, the steps of which have been followed by Chen (1984) to carry out the evaluation with an additional harmonic oscillator potential, a case treated also by Castrigiano and Kokiantonis (1983).

The present work places emphasis on auxiliary fields for handling two-time actions. In the literature there have been designed various forms of functional integrals over quite general fields including temporal as well as spatial dependence. The reader is referred to Edwards and Peierls (1954), Matthews and Salam (1955), Edwards (1965), Edwards and Sherrington (1967) and Sherrington (1971). For functional integrals over fields arising in polymers see Edwards and Freed (1970). For functional integrals in fluid mechanics see Rosen (1983). However, here we rely on fields with only temporal dependence and in this sense there is some resemblance to earlier works by the author (Papadopoulos 1968, 1974).

In § 2 complex and real fields are introduced and certain general transformations converting path into functional integrals over these fields are obtained. Hereafter the term path integration will be used whenever the integration is over particle coordinates while the term functional integration will be employed in relation to other fields. Utilising a complex field we treat explicitly a particular, but rather general, form of a quadratic action. Finally, the propagator for a specific quartic action is reduced to an integral over a single variable.

In § 3 we present a method which employs a dynamical auxiliary field enabling the handling of cases of quadratic actions. It is shown how the method reproduces the various explicitly known results, and furthermore how to generate explicitly certain new ones. By way of an example a further path integral is treated.

In an appendix we work out a new, as far as we are aware of, functional integral identity.

2. Real and complex fields

We begin by considering the two-time action

$$S[x(t)] = \int_0^T \mathrm{d}t\, \tfrac{1}{2} m \dot{x}^2(t) - \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, W(x(t) - x(s)) \tag{2.1}$$

where W may explicitly depend on t and s. The propagator, K, associated with the action (2.1) is given by the Feynman path integral

$$K(xT|x'0)=\int_{x(0)=x',x(T)=x}\exp\left(\frac{\mathrm{i}}{\hbar}S[x(t)]\right)\mathscr{D}[x]. \tag{2.2}$$

The path integral (2.2) can be written with the aid of a complex auxiliary field, $\lambda(t)$, as follows:

$$\begin{aligned}K(xT|x'0)=\int_{x(0)=x',x(T)=x}&\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\frac{m}{2}\dot{x}^2(t)\right.\\&+\int_0^T \mathrm{d}t\,\lambda^*(t)x(t)-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, W(\lambda(t)-\lambda(s))\\&\left.-\int_0^T \mathrm{d}t\,|\lambda(t)|^2\right]\mathscr{D}[x]\prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right)\mathrm{d}^2\lambda(t)\end{aligned} \tag{2.3}$$

where $\mathrm{d}^2\lambda(t)$ stands as per usual for $\mathrm{d}(\mathrm{Re}\,\lambda(t))\mathrm{d}(\mathrm{Im}\,\lambda(t))$. The integration over the complex field, $\lambda(t)$, has the effect of replacing $\lambda(t)$ by $x(t)$, the coefficient of $\lambda^*(t)$. The operation is making use of a δ functional. For details see the appendix.

The path integral in (2.3) over $x(t)$ can now be performed, and it is the case of a particle in a time-prescribed field of force $F(t)=-\mathrm{i}\hbar\lambda^*(t))$. Utilising Feynman and Hibbs (1965) we arrive at

$$\begin{aligned}K(xT|x'0)=&\left(\frac{m}{2\pi\mathrm{i}\hbar T}\right)^{1/2}\exp\left(\frac{\mathrm{i}m}{2\hbar T}(x-x')^2\right)\\&\times\int\exp\left[\frac{1}{2}\int_0^T \mathrm{d}t\,\lambda^*(t)\left(x'+\frac{x-x'}{T}t\right)-\int_0^T \mathrm{d}t\,|\lambda(t)|^2\right.\\&\left.+\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\left(\frac{\hbar^2}{2}C(t,s)\lambda^*(t)\lambda^*(s)-W(\lambda(t)-\lambda(s))\right)\right]\\&\times\prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right)\mathrm{d}^2\lambda(t)\end{aligned} \tag{2.4}$$

where $C(t,s)$ is a symmetric function given by

$$C(t,s)=\frac{1}{m}\left(\frac{t+s}{2}-\frac{ts}{T}-\frac{|t-s|}{2}\right). \tag{2.4a}$$

Let us now introduce the functional integral identity

$$\begin{aligned}&\int\exp\left(\Psi[\lambda^*(t)]+\int_0^T \mathrm{d}t\,\lambda^*(t)R(t)+\Phi[\lambda(t)]-\int_0^T \mathrm{d}t\,|\lambda(t)|^2\right)\prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right)\mathrm{d}^2\lambda(t)\\&\quad=\int\exp\left(\Psi[\lambda^*(t)]+\Phi[\lambda(t)+R(t)]-\int_0^T \mathrm{d}t\,|\lambda(t)|^2\right)\\&\qquad\times\prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right)\mathrm{d}^2\lambda(t).\end{aligned} \tag{2.5}$$

The functionals $\Psi[\lambda^*(t)]$ and $\Phi[\lambda(t)]$ in the integrand of (2.5) depend respectively on $\lambda^*(t)$ and $\lambda(t)$ to the exclusion of their conjugates. Furthermore, the functionals $\exp(\Phi[\lambda(t)])$ and $\exp(\Psi[\lambda(t)])$ are either integral functionals of $\lambda(t)$, or if composed of a series of functionals they are termwise integral in which case the functional integrations precede the series summation. A discussion concerning this sort of procedure is given in the appendix for the case of functions integrated against a δ function. The derivation of (2.5) is given in the appendix.

Upon application of (2.5) to (2.4) we have:

$$K(xT|x'0) = K_0(xT|x'0) \times \int \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \left(\frac{\hbar^2}{2} C(t,s)\lambda^*(t)\lambda^*(s) - W(\lambda(t)-\lambda(s)) + \frac{x-x'}{T}(t-s)\right) - \int_0^T \mathrm{d}t\, |\lambda(t)|^2\right] \prod_{0\leq t<T} \left(\frac{\mathrm{d}t}{\pi}\right) \mathrm{d}^2\lambda(t) \tag{2.6}$$

where in (2.6) we have made use of the notation $K_0(xT|x'0)$ for the free particle propagator preceding the integral symbol in (2.4). Equation (2.6) is a functional integral over a complex field giving the propagator associated with the action (2.1). Clearly (2.6) shows that the propagator, regarding its spatial dependence, is a function of $(x-x')$, a property relating to the translation invariance of the action (2.1).

Let us now express the propagator $K(xT|x'0)$ as a functional integral over a real field $q(t)$. This is easily attained using (2.6) as follows

$$K(xT|x'0) = K_0(xT|x'0) \times \int \exp\left(\frac{\mathrm{i}}{2\hbar}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, C'(t,s)q(t)q(s) + \int_0^T \mathrm{d}t\, \lambda^*(t)q(t) - \int_0^T \mathrm{d}t\, |\lambda(t)|^2 - \frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, W(\lambda(t)-\lambda(s)) + \frac{x-x'}{T}(t-s)\right) \times [\det 2\pi\mathrm{i}\hbar C'(t,s)]^{-1/2} \prod_{0<t<T} \mathrm{d}q(t) \prod_{0\leq t<T} \left(\frac{\mathrm{d}t}{\pi}\right) \mathrm{d}^2\lambda(t) \tag{2.7}$$

where $C'(t,s)$ in (2.7) is the inverse of $C(t,s)$ i.e.

$$\int_0^T \mathrm{d}\sigma\, C(t,\sigma)C'(\sigma,s) = \delta(t-s). \tag{2.7a}$$

Equation (2.7) is seen to be valid since integration over $q(t)$ will generate in the exponent the expression $(\mathrm{i}\hbar/2)\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, C(t,s)\lambda^*(t)\lambda^*(s)$. Considering that $\partial^2 C(t,s)/\partial\sigma^2 = \delta(t-\sigma)/m$ there follows that the inverse matrix can be written explicitly as

$$C'(t,s) = m(\partial^2/\partial t^2)\delta(t-s). \tag{2.7b}$$

The integration over $q(t)$ does not include $q(0)$ among the integration variables on account of $C(0,s) = C(t,0) = 0$.

Upon integrating over the complex field $\lambda(t)$ in (2.7) $q(t)$ will replace $\lambda(t)$ in W and our propagator takes the form of a functional integral over a real field as

$$K(xT|x'0) = K_0(xT|x'0) \times \int \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \left[\tfrac{1}{2}C'(t,s)q(t)q(s)\right.\right.$$

$$- W\left(q(t)-q(s)+\frac{x-x'}{T}(t-s)\right)\bigg]\bigg\}$$
$$\times[\det 2\pi i\hbar C(t,s)]^{-1/2} \prod_{0<t<T} \mathrm{d}q(t). \tag{2.8}$$

Equation (2.8) could also have been obtained by use of the real form of the δ functional, namely

$$\prod_t \delta(x(t)-q(t)) = \int \exp\left(\mathrm{i}\int_0^T \mathrm{d}t\, k(t)(x(t)-q(t))\right) \prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right) \mathrm{d}K(t) \tag{2.9}$$

in conjunction with the original path integral. For use of the real δ functional (2.9) elsewhere see Hosokawa (1967).

Tarski (1968) gave certain functional integral transformations over real fields for the Wiener integral, mainly in the unconditional case. The complex field transformations and in particular the functional integral identity (2.5) we believe to be new.

A more general expression for W has the form $W(x(t), x(s))$ with explicit dependence on t and s. In this case the expression for W in (2.7) will be $W(\lambda(t)+x'+(x-x')t/T, \lambda(s)+x'+(x-x')s/T)$ and correspondingly the W for (2.8) will be obtained from the above replacing $\lambda(t)$, $\lambda(s)$ by $q(t)$, $q(s)$.

Taking $W = G(t,s)x(t)x(s)$, which belongs to the above case, one can proceed with the aid of (2.8) to re-establish the Adamowski *et al* (1982) result.

As a further application consider the memory action

$$S_1[x(t)] = \int_0^T \mathrm{d}t \frac{m}{2}\dot{x}^2(t) - \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\,[a(t)b(s)+a(s)b(t)]x(t)x(s)$$
$$= \int_0^T \mathrm{d}t \frac{m}{2}\dot{x}^2(t) - \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, a(t)b(s)x(t)x(s). \tag{2.10}$$

Use of the complex auxiliary field as in (2.3) leads to the functional integral below for the propagator associated with the action (2.10)

$$K_1(xT|x'0) = K_0(xT|x'0)$$
$$\times \int \exp\bigg[\tfrac{1}{2}\int_0^T \mathrm{d}t\, X_0(t)\lambda^*(t) + \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s$$
$$\times\left(\frac{\mathrm{i}\hbar}{2} C(t,s)\lambda^*(t)\lambda^*(s) - \frac{\mathrm{i}}{\hbar} a(t)b(s)\lambda(t)\lambda(s)\right)$$
$$- \int_0^T \mathrm{d}t\, |\lambda(t)|^2\bigg] \prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right) \mathrm{d}^2\lambda(t). \tag{2.11}$$

At this stage we introduce the notation

$$\int_0^T \mathrm{d}t\, a(t)X_0(t) = (aX_0), \qquad \int_0^T \mathrm{d}t\, b(t)X_0(t) = (bX_0)$$
$$\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, C(t,s)(a(t)a(s), a(t)b(s), b(t)b(s))$$
$$= ((aCa), (aCb), (bCb)) \tag{2.12}$$

to be used in the process of evaluation of (2.11).

A further complex auxiliary variable, z, enables us to replace the bilinear functional in (2.11) by a linear functional in $\lambda(t)$ as follows

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, a(t)b(s)\lambda(t)\lambda(s)\right)$$
$$= \int \exp\left[\mathrm{i}\int_0^T \mathrm{d}t \left(\frac{z}{\sqrt{\hbar}}a(t)+\frac{z^*}{\sqrt{\hbar}}b(t)\right)\lambda(t)+\mathrm{i}|z|^2\right]\frac{\mathrm{d}^2 z}{\mathrm{i}\pi}. \tag{2.13}$$

With this substitution in (2.11) the integration over $\lambda(t)$ can now be performed, resulting in transmitting $-(za(t)+z^*b(t))/\sqrt{\hbar}$ in place of $\lambda^*(t)$, and we obtain

$$K_1(xT|x'0) = K_0(xT|x'0)$$
$$\times \int \exp\left(-\frac{1}{2\sqrt{\hbar}}[(aX_0)z+(bX_0)z^*]\right.$$
$$\left.-\frac{\mathrm{i}}{2}[(aCa)z^2+(bCb)z^{*2}]-\mathrm{i}[(aCb)-1]|z|^2\right)\frac{\mathrm{d}^2 z}{\mathrm{i}\pi} \tag{2.14}$$

where in (2.14) we have made use of the notation (2.12) and have furthermore used $(bCa)=(aCb)$.

Linearisation of the term involving z^2 in (2.14) via the transformation

$$\exp\left(-\frac{\mathrm{i}}{2}(aCa)z^2\right) = \int \exp[\xi z - \mathrm{i}\xi^2/2(aCa)]\frac{\mathrm{d}\xi}{[-2\pi\mathrm{i}(aCa)]^{1/2}} \tag{2.15}$$

enables us to integrate over z, and subsequent integration over ξ leads to the desired propagator in an explicit form, as

$$K_1(xT|x'0) = K_0(xT|x'0)(\mathrm{i}/D^{1/2})\exp\left(-\frac{\mathrm{i}}{8\hbar D}\{(bCb)(aX_0)^2+2[1-(aCb)]\right.$$
$$\left.\times(aX_0)(bX_0)+(aCa)(bX_0)^2\}\right) \tag{2.16}$$

where

$$D=(aCa)(bCb)-[(aCb)-1]^2. \tag{2.16a}$$

Result (2.15) could have also been obtained by use of the real form (2.9) of the δ functional.

As a further example let us consider the quartic action

$$S_2[x(t)] = \frac{m}{2}\int_0^T \mathrm{d}t\,(\dot{x}^2(t)-\omega^2 x^2(t)) - \frac{m^2 w^2}{4}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, x^2(t)x^2(s). \tag{2.17}$$

Making use of the identity

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\frac{m^2 w^2}{4}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, x^2(t)x^2(s)\right)$$
$$= \int \exp\left(-\frac{\mathrm{i}}{\hbar}\frac{m}{2}wg\int_0^T \mathrm{d}t\, x^2(t) - \frac{g^2}{4\mathrm{i}\hbar}\right)\frac{\mathrm{d}g}{(4\pi\mathrm{i}\hbar)^{1/2}} \tag{2.18}$$

in the path integral giving the propagator K_2 associated with the action (2.17) we arrive at the equation

$$K_2(xT|x'0)=\int_{x(0)=x',x(T)=x}\exp\left(\frac{\mathrm{i}}{\hbar}S_3[x(t)]\right)\mathscr{D}[x]$$
$$=\int\frac{\mathrm{d}g}{(4\pi\mathrm{i}\hbar)^{1/2}}\exp\left(-\frac{g^2}{4\mathrm{i}\hbar}\right)$$
$$\times\int_{x(0)=x',x(T)=x}\exp\left(\frac{\mathrm{i}m}{2\hbar}\int_0^T\mathrm{d}t\,[\dot{x}^2(t)-(\omega^2+wg)x^2(t)]\right)\mathscr{D}[x]. \qquad (2.19)$$

The path integral on the far RHS of (2.19) can be evaluated from the formula for the propagator of an oscillator (see Feynman and Hibbs 1965), and thus the required propagator takes the form

$$K_2(xT|x'0)=\int\frac{\mathrm{d}g}{(4\pi\mathrm{i}\hbar)^{1/2}}\exp\left(-\frac{g^2}{4\mathrm{i}\hbar}\right)\left(\frac{m\Omega_g}{2\pi\mathrm{i}\hbar\sin\Omega_gT}\right)^{1/2}$$
$$\times\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m\Omega_g}{2\sin\Omega_gT}[(x^2+x'^2)\cos\Omega_gT-2xx']\right) \qquad (2.20)$$

where

$$\Omega_g=(\omega^2+wg)^{1/2}. \qquad (2.20a)$$

On replacing ω^2 by $-\omega^2$ in the action (2.17) there emerges a good deal of similarity with the action

$$\frac{m}{2}\int_0^T\mathrm{d}t\,(\dot{x}^2(t)+\omega^2x^2(t)-\tfrac{1}{2}mw^2x^4(t))$$

used in a model for phase transitions. Because of this the above evaluation may have some usefulness in phase transition studies.

3. Dynamical fields

As pointed out in the introduction the effect of eliminating the degrees of freedom of a system 'Q', under certain conditions, coupled to a system 'X' manifests itself in a new Lagrangian of 'X', viewed as a separate system, in the form of a memory potential function. A particular case of this nature appears in Osaka (1959), but there he did not proceed to extract the full expression for the related propagator. In this section we shall exploit the above situation for the purpose of treating propagators relating to certain quadratic actions which have recently appeared in the literature (Adamowski *et al* 1982, Khandekar *et al* 1983, Castrigiano and Kokiantonis 1983, 1984) through a unified formula involving evaluations with single-time Lagrangians. We shall furthermore show how to generate certain new results.

In order to demonstrate the procedure we consider a combined system 'X-Q' with Lagrangian

$$L=L_0+L_Q+U_\mathrm{I} \qquad (3.1)$$

where L_0 is a single-time general quadratic Lagrangian involving only the coordinates

$\boldsymbol{x}$ of 'X', and

$$L_Q = \tfrac{1}{2}m(\dot{\boldsymbol{q}}^2 - \omega^2 \boldsymbol{q}^2), \qquad U_I = mk\boldsymbol{q}\cdot\boldsymbol{x}. \tag{3.1a}$$

U_I is an interaction potential energy between the oscillator 'Q' and the system 'X'. It should be noted that the range of values of the coupling constant k is restricted if the system is required to remain in a finite region.

The propagator associated with the Lagrangian L can be written in the form of a Feynman path integral over the $\boldsymbol{x}$ and $\boldsymbol{q}$ paths

$$K(\boldsymbol{x}\boldsymbol{q}T|\boldsymbol{x}'\boldsymbol{q}'0) = \int_{\boldsymbol{x}(0)=\boldsymbol{x}',\boldsymbol{x}(T)=\boldsymbol{x};\boldsymbol{q}(0)=\boldsymbol{q}',\boldsymbol{q}(T)=\boldsymbol{q}} \times \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^T (L_0[\boldsymbol{x}(t)] + L_Q[\boldsymbol{q}(t)] + mk\boldsymbol{q}(t)\cdot\boldsymbol{x}(t))\,\mathrm{d}t\right)\mathscr{D}[\boldsymbol{x}]\mathscr{D}[\boldsymbol{q}]. \tag{3.2}$$

Upon performing the $\boldsymbol{q}$ path integration in (3.2) we arrive at

$$K(\boldsymbol{x}\boldsymbol{q}T|\boldsymbol{x}'\boldsymbol{q}'0) = \left(\frac{m\omega}{2\pi \mathrm{i}\hbar \sin\omega T}\right)^{3/2} \times \int_{\boldsymbol{x}(0)=\boldsymbol{x}',\boldsymbol{x}(T)=\boldsymbol{x}} \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^T L_0[\boldsymbol{x}(t)]\,\mathrm{d}t + \frac{\mathrm{i}}{\hbar}S_{QI}(\boldsymbol{q}T|\boldsymbol{q}'0;[\boldsymbol{x}(t)])\right)\mathscr{D}[\boldsymbol{x}] \tag{3.3}$$

where S_{QI} in (3.3) is the classical action relating to the Lagrangian $L_Q + U_I$, and is a functional of $\boldsymbol{x}(t)$. It is given by

$$\begin{aligned} S_{QI}(\boldsymbol{q}T|\boldsymbol{q}'0;[\boldsymbol{x}(t)]) &= \frac{m\omega}{2\sin\omega T}[(\boldsymbol{q}^2 + \boldsymbol{q}'^2)\cos\omega T - 2\boldsymbol{q}\cdot\boldsymbol{q}'] \\ &\quad + \frac{mk}{\sin\omega T}\int_0^T \mathrm{d}t\,[\boldsymbol{q}'\sin\omega(T-t) + \boldsymbol{q}\sin\omega t]\cdot\boldsymbol{x}(t) \\ &\quad - \frac{mk^2}{\omega\sin\omega T}\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\,\sin\omega(T-t)\sin\omega s\,\boldsymbol{x}(t)\cdot\boldsymbol{x}(s). \end{aligned} \tag{3.3a}$$

The path integral of the RHS of (3.3) relates to a two-time quadratic Lagrangian. It is also parametrised by the end conditions obeyed by the $\boldsymbol{q}$ paths. If we choose to eliminate the $\boldsymbol{q}$ end conditions in various ways, e.g. integrating (3.3) against a weight for $\boldsymbol{q}$, $\boldsymbol{q}'$, we end up generating different memory actions.

Let a particular weight be $M(\boldsymbol{q}, \boldsymbol{q}')$. Integrating (3.3) against M we obtain an identity involving a certain type of a memory path integral.

$$\begin{aligned} &\int\left[\int \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^T L_0[\boldsymbol{x}(t)]\,\mathrm{d}t + \frac{\mathrm{i}}{\hbar}S_{QI}(\boldsymbol{q}T|\boldsymbol{q}'0;[\boldsymbol{x}(t)])\right)M(\boldsymbol{q},\boldsymbol{q}')\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{q}'\right]\mathscr{D}[\boldsymbol{x}] \\ &\quad = \left(\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega T}\right)^{-3/2}\left(\det\frac{\mathrm{i}}{2\pi\hbar}\frac{\partial^2 S(\boldsymbol{x}\boldsymbol{q}T|\boldsymbol{x}'\boldsymbol{q}'0)}{\partial(\boldsymbol{x},\boldsymbol{q})\partial(\boldsymbol{x}',\boldsymbol{q}')}\right)^{1/2} \\ &\quad\quad \times\int \exp\left(\frac{\mathrm{i}}{\hbar}S(\boldsymbol{x}\boldsymbol{q}T|\boldsymbol{x}'\boldsymbol{q}'0)\right)M(\boldsymbol{q},\boldsymbol{q}')\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{q}' \end{aligned} \tag{3.4}$$

where $S(\boldsymbol{x}\boldsymbol{q}T|\boldsymbol{x}'\boldsymbol{q}'0)$ is the classical action relating to the Lagrangian $L_0 + L_Q + U_I$ from the spacetime point $(\boldsymbol{x}', \boldsymbol{q}', 0)$ to the spacetime point $(\boldsymbol{x}, \boldsymbol{q}, T)$.

For the derivation of (3.4) we have evaluated the propagator K in (3.3) noting that it relates to a single-time quadratic Lagrangian, (3.1), and therefore can be obtained by use of the classical action, S, associated with the Lagrangian (3.1). The evaluation proceeds (as in Jones and Papadopoulos (1971) for a particle in an homogeneous magnetic field) using a result originally due to Van Vleck (1928). It should be noted here that this procedure does not apply whenever memory is present in the Lagrangian.

Formula (3.4) constitutes the main result of this section. It provides fully the time dependent pre-exponential factor for the propagator associated with the memory Lagrangian generated by the integration over $\boldsymbol{q}$, $\boldsymbol{q}'$ on the LHS of (3.4). The approach covers a wide range of a certain type of memory Lagrangian and can go beyond quadratic Lagrangians by appropriate choice of the weight M. Nevertheless, its applicability remains rather limited.

By appropriate choice of L_0, k and $M(\boldsymbol{q}, \boldsymbol{q}')$ the evaluations given in the references at the start of this section can be obtained. For instance the case of Khandekar *et al* (1983) requires taking $k=\Omega\omega$ and $M=\delta(\boldsymbol{q}-\boldsymbol{q}')$. However, rather than reproduce known results it would be more profitable to invest space on a certain new evaluation.

Let us set in (3.4) $M=\delta(\boldsymbol{q})\delta(\boldsymbol{q}')$ and thus obtain in the exponential argument on the LHS of (3.4) the Lagrangian

$$L'=L_0-\frac{mk^2}{\omega \sin \omega T}\int_0^t \mathrm{d}s \sin \omega(T-t) \sin \omega s \boldsymbol{x}(t)\cdot\boldsymbol{x}(s). \tag{3.5}$$

Let us also take

$$L_0=\tfrac{1}{2}m(\dot{\boldsymbol{x}}^2-\Omega^2\boldsymbol{x}^2). \tag{3.5a}$$

For obtaining the corresponding S related to the Lagrangian $L=L_0+L_{\mathrm{Q}}+U_{\mathrm{I}}$ we require the roots of the secular equation associated with the classical equation of motion deriving from L, which is:

$$\rho^4-(\Omega^2+\omega^2)\rho^2+(\Omega^2\omega^2-k^2)=0. \tag{3.5b}$$

In order to guarantee finiteness in the system's motion (also periodicity) it is necessary to restrict k in the range $k\leqslant\Omega\omega$.

The roots of (3.7b) are given by $\rho=\pm\nu_1$, $\rho=\pm\nu_2$, where

$$\nu_1=\left\{\frac{\Omega^2+\omega^2}{2}+\left[\left(\frac{\Omega^2-\omega^2}{2}\right)^2+k^2\right]^{1/2}\right\}^{1/2},$$

$$\nu_2=\left\{\frac{\Omega^2+\omega^2}{2}-\left[\left(\frac{\Omega^2-\omega^2}{2}\right)^2+k^2\right]^{1/2}\right\}^{1/2}. \tag{3.5c}$$

The action along the classical path from $(\boldsymbol{x}', \boldsymbol{q}', 0)$ to $(\boldsymbol{x}, \boldsymbol{q}, T)$ is given by

$$S(\boldsymbol{x}\boldsymbol{q}T|\boldsymbol{x}'\boldsymbol{q}'0)=\frac{m}{2(\nu_1^2-\nu_2^2)}\bigg[(\boldsymbol{x}^2+\boldsymbol{x}'^2)[(\nu_1^2-\omega^2)\nu_1\cot\nu_1T-(\nu_2^2-\omega^2)\nu_2\cot\nu_2T]$$
$$-2\boldsymbol{x}\cdot\boldsymbol{x}'\left(\frac{(\nu_1^2-\omega^2)\nu_1}{\sin\nu_1T}-\frac{(\nu_2^2-\omega^2)\nu_2}{\sin\nu_2T}\right)$$
$$+(\boldsymbol{q}^2+\boldsymbol{q}'^2)[(\nu_1^2-\omega^2)\nu_2\cot\nu_2T-(\nu_2^2-\omega^2)\nu_1\cot\nu_1T]$$
$$-2\boldsymbol{q}\cdot\boldsymbol{q}'\left(\frac{(\nu_1^2-\omega^2)\nu_2}{\sin\nu_2T}-\frac{(\nu_2^2-\omega^2)\nu_1}{\sin\nu_1T}\right)$$

$$+2(\boldsymbol{q}\cdot\boldsymbol{x}+\boldsymbol{q}'\cdot\boldsymbol{x}')k(\nu_1\cot\nu_1T-\nu_2\cot\nu_2T)$$
$$-2(\boldsymbol{q}\cdot\boldsymbol{x}'+\boldsymbol{q}'\cdot\boldsymbol{x})k\left(\frac{\nu_1}{\sin\nu_1T}-\frac{\nu_2}{\sin\nu_2T}\right)\bigg]. \tag{3.5d}$$

Inserting now (3.5d) into (3.6) we obtain the following explicit path integral evaluation:

$$\int_{\boldsymbol{x}(0)=\boldsymbol{x}',\boldsymbol{x}(T)=\boldsymbol{x}}\exp\left(\frac{\mathrm{i}}{\hbar}\int_0^T\mathrm{d}t\,L'[\boldsymbol{x}(t)]\right)\mathscr{D}[\boldsymbol{x}]=\left(\frac{m\Omega^2(\omega^2\Omega^2-k^2)}{\pi\mathrm{i}\hbar\omega(\nu_1^2-\nu_2^2)^2}\frac{\sin\omega T}{\sin\nu_1T\sin\nu_2T}\right)^{3/2}$$
$$\times\exp\left\{\frac{\mathrm{i}m}{2\hbar(\nu_1^2-\nu_2^2)}\Big[(\boldsymbol{x}^2+\boldsymbol{x}'^2)[(\nu_1^2-\omega^2)\nu_1\cot\nu_1T-(\nu_2^2-\omega^2)\nu_2\cot\nu_2T]\right.$$
$$\left.-2\boldsymbol{x}\cdot\boldsymbol{x}'\left((\nu_1^2-\omega^2)\frac{\nu_1}{\sin\nu_1T}-(\nu_2^2-\omega^2)\frac{\nu_2}{\sin\nu_2T}\right)\Big]\right\}. \tag{3.6}$$

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Appendix

The real δ function operation involves a particular real variable function $\delta(x, R)$ whose integration against a function $f(x)$ transmits to the argument of f the value R. As is well known the operation reads:

$$\int f(x)\delta(x, R)\,\mathrm{d}x=f(R).$$

A situation of this nature with the role of the variable x taken up by a complex variable $z=x+\mathrm{i}y$ can exist; the integration now being performed over the real and imaginary parts of z.

A representation of the δ function in the above complex case is given by

$$\Delta(z, R)\,\mathrm{d}^2z=\exp(Rz^*-|z|^2)\,\mathrm{d}^2z/\pi \tag{A1}$$

and the function f has to be expressed explicitly as a function of z only.

To enable ourselves to proceed along these lines we consider the integral.

$$\int\exp(\xi z)\exp(Rz^*-|z|^2)\,\mathrm{d}^2z/\pi=\exp(\xi R) \tag{A2}$$

which is seen to hold utilising the Cartesian form of the variable z after performing the resulting Gaussian integrations. ξ and R can in general be complex. For other applications of such integrations the reader is refered to Mühlschlegel (1978). Notice that (A1) in (A2) acts as a δ function. The same applies for a linear combination of Fourier exponentials, $\exp(k_nz)$.

Next, if $f(z)$ is an integral function the identity

$$I_0 = \int f(z) \exp(Rz^* - |z|^2)\, d^2z/\pi = f(R) \tag{A3}$$

holds, for f can be represented through a power series as

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

and therefore I_0 can be generated as

$$I_0 = \sum_{n=0}^{\infty} f_n \frac{\partial^n}{\partial \xi^n} \exp(\xi R)|_{\xi=0} = f(R).$$

The δ function character of (A1) on integration with a rapidly growing function as $|z| \to \infty$ overrides convergence difficulties through cancellations produced by the combined integrations over Re z and Im z. Thus, e.g. we have

$$\int \exp(z^4 + Rz^* - |z|^2) \frac{d^2z}{\pi} = \exp(R^4)$$

a convergent result, in spite of the occurrence of a factor $\exp(x^4)$ in the integrand.

We can further consider the integration of $g(z^*)f(z)$ against (A1) with $g(z)$ also an integral function and therefore given as

$$g(z) = \sum_{n=0}^{\infty} g_n z^n.$$

Since $g(z^*) \exp(Rz^*)$ can be generated as

$$g(z^*) \exp(Rz^*) = g(\partial/\partial R) \exp(Rz^*)$$

we have with the aid of (A3) the result

$$\int g(z^*) f(z) \exp(Rz^* - |z|^2)\, d^2z/\pi = g(\partial/\partial R) f(R). \tag{A4}$$

(A3) is also valid when z and z^* exchange positions. Under this exchange a particular case of (A4) with $g(z) = z$ appeared earlier (Papadopoulos 1980).

In the above cases the δ function character of (A1) is rigorously exhibited. Nevertheless, these instances do not exhaust all δ function operations which one may encounter in applications. However, as with the real δ function one may be faced with convergence flaws. In the real case the difficulties reside in the δ function expansions. While expression (A1) is free of such subtleties it would seem inescapable that they should disappear completely from the complex treatment. We shall see that, once outside the regime of integral functions, convergence difficulties make their way through the expansion used to represent the transcription of the function $f(x)$ to the complex regime. In both cases, real and complex, these procedures, involving a faulty step, provide through the integrations a reliable final product. The likelihood of developing a theory for the complex case in compliance with present day mathematical standards of rigour exists. However, the exposition here cannot lay claim on a level of presentation beyond the sort accepted in most physical treatments.

In the spirit of the above paragraph we proceed with treating the case when $f(x) = |x|^{-1}$. In order to employ (A3), even formally, we require an explicit function of z and only of z leading to $|x|^{-1}$ for $z=x$. Evidently $|z|^{-1}$ must be excluded since it is also a function of z^*. However, the transform

$$2\pi^{1/2}\int_0^\infty \exp(-\xi^2x^2)\,d\xi=|x|^{-1}$$

provides the required combination, which is an integral of a function of x that can be extended to the z plane and be an explicit function of z only. Each of the functions $\exp(-\xi^2z^2)$ is an integral function of z, and for which, therefore, (A3) applies perfectly. On the other hand the integral over ξ from 0 to ∞ does not converge for all z. Nevertheless, the following is true:

$$2\pi^{1/2}\int_0^\infty d\xi\int \exp(-\xi^2z^2)\exp(Rz^*-|z|^2)\frac{d^2z}{\pi}=|R|^{-1}$$

showing a procedure of applying the δ function operation in this case.

The important case of the Coulomb potential can be handled via the transform

$$(x_1^2+x_2^2+x_3^2)^{-1/2}=\frac{1}{2\pi^2}\int\frac{d\boldsymbol{k}}{k^2}\exp[i(k_1x_1+k_2x_2+k_3x_3)]$$

where $k^2=k_1^2+k_2^2+k_3^2$, but here we require a three-variable z_1, z_2, z_3 complex δ function.

Having dealt to some extent with a complex form of the δ function operation we can now with relatively little effort transcribe the operation to the functional regime. A δ functional is essentially a 'product of δ functions' over all the variables involved in the argument of the functional against which the δ functional operation is to be applied. Since in general the range of indices labelling the various variables is a continuous interval, say $0\leq t<T$, we can associate a 'dt' per variable and the δ functional will read

$$\mathscr{D}([z(t)],[R(t)])\prod_{0\leq t<T}d^2z(t)=\exp\left(\int_0^T dt(R(t)z^*(t)-|z(t)|^2)\right)\prod_{0\leq t<T}\left(\frac{dt}{\pi}\right)d^2z(t) \tag{A5}$$

(A5) when integrated with a functional $\mathscr{F}[z(t)]$ which is an explicit functional of $z(t)$ only will transmit in place of $z(t)$ the value $R(t)$. Thus, we have

$$\int\mathscr{F}[z(t)]\exp\left(\int_0^T dt\,R(t)z^*(t)-\int_0^T dt\,|z(t)|^2\right)\prod_{0\leq t<T}\left(\frac{dt}{\pi}\right)d^2z(t)=\mathscr{F}[R(t)]. \tag{A6}$$

Let us now proceed to show (2.5). Under the conditions stated about Ψ and Φ in the text we can write with the aid of (A6).

$$\exp(\Psi[\lambda^*(t)])=\int\exp(\Psi[z^*(t)])\exp\left(\int_0^T dt\,\lambda^*(t)z(t)-\int_0^T dt\,|z(t)|^2\right)$$
$$\times\prod_{0\leq t<T}\left(\frac{dt}{\pi}\right)d^2z(t). \tag{A7}$$

Notice that in (A7) we have exchanged $z(t)$ with $z^*(t)$ as this does not affect the role of the δ functional operation, consisting in replacing $z^*(t)$ in Ψ by the coefficient of $z(t)$, $\lambda^*(t)$, in the linear term of the exponential on the LHS of (A7)

Denoting by I the LHS of (2.5) and utilising (A7) we have

$$I=\int \exp\left(\Psi[z^*(t)]+\int_0^T \mathrm{d}t\,\lambda^*(t)(R(t)+z(t))+\Phi[\lambda(t)]-\int_0^T \mathrm{d}t(|\lambda(t)|^2+|z(t)|^2)\right)$$
$$\times \prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right)^2 \mathrm{d}^2z(t)\,\mathrm{d}^2\lambda(t). \tag{A8}$$

Now performing the integration over all $\lambda(t)$ we have

$$I=\int \exp(\Psi[z^*(t)]+\Phi[z(t)+R(t)]-\int_0^T \mathrm{d}t\,|z(t)|^2)\prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right)\mathrm{d}^2z(t). \tag{A9}$$

Considering that $z(t)$ is a dummy variable it can be replaced by $\lambda(t)$ and so the identity (2.5) is now established.

We can now go a step further. On taking in (A9) the McLaurin expansion of $\Psi[z^*(t)]$ by virtue of (2.5) the coefficient of the linear term in $z^*(t)$ will be added to the argument of Φ and we have

$$\int \exp\left(\Psi[z^*(t)]+\Phi[z(t)]+\int_0^T \mathrm{d}t\,R(t)z^*(t)-\int_0^T \mathrm{d}t\,|z(t)|^2\right)\prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right)\mathrm{d}^2z(t)$$
$$=\int \exp\left\{\Psi[z^*(t)]-\int_0^T \mathrm{d}t\left(\frac{\delta\Psi}{\delta z^*(t)}\right)_{z^*=0} z^*(t)+\Phi\left[z(t)+R(t)\right.\right.$$
$$\left.\left.+\left(\frac{\delta\Psi}{\delta z^*(t)}\right)_{z^*=0}\right]-\int_0^T \mathrm{d}t\,|x(t)|^2\right\}\prod_{0\leq t<T}\left(\frac{\mathrm{d}t}{\pi}\right)\mathrm{d}^2z(t). \tag{A10}$$

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